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# Stochastic Perturbation of Power Law Optical Solitons

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The soliton perturbation theory is used to study and analyze the stochastic perturbation of optical solitons, with power law nonlinearity, in addition to deterministic perturbations, that is governed by the nonlinear Schrödinger's equation. The Langevin equations are derived and analysed. The deterministic perturbations that are considered here are due to filters and nonlinear damping.

**KEY WORDS:** optical solitons; power law; soliton perturbation; Langevin equations.

**OCIS** Codes: 060.2310; 060.4510; 060.5530; 190.4370.

## 1. INTRODUCTION

The dynamics of pulses propagating in optical fibers has been a major area of research given its potential applicability in all optical communication systems. It has been well established (Hasegawa and Kodama, 1995; Wabnitz *et al.*, 1995) that this dynamics is described, to first approximation, by the integrable Nonlinear Schrödinger Equation (NLSE). Here the global characteristics of the pulse envelope can be fully determined by the method of Inverse Scattering Transform (IST) and in many instances, the interest is restricted to the single pulse described by the one soliton form of the NLSE. Typically though, distortions of these pulses arise due to perturbations which are either higher order corrections in the model as derived from the original Maxwell's equations (Hasegawa and Kodama, 1995; Wabnitz *et al.*, 1995), physical mechanisms not considered at first approximation like Raman effects or external perturbations such as the lumped effect due to the addition of bandwidth limited amplifiers in a communication line. Mathematically, these corrections are seen as perturbations of the NLSE and most of them

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have been studied thoroughly by regular asymptotic (Biswas, 2003), soliton perturbation (Hasegawa and Kodama, 1995) or Lie tranform (Hasegawa and Kodama, 1995) methods.

Besides the deterministic type perturbations one also needs to take into account, from practical considerations, the stochastic type perturbations. These effects can be classified into three basic types:

1. Stochasticity associated with the chaotic nature of the initial pulse due to partial coherence of the laser generated radiation.
2. Stochasticity due to random nonuniformities in the optical fibers like the fluctuations in the values of dielectric constant the random variations of the fiber diameter and more.
3. The chaotic field caused by a dynamic stochasticity might arise from a periodic modulation of the system parameters or when a periodic array of pulses propagate in a fiber optic resonator.

Thus, stochasticity is inevitable in optical soliton communications (Abdullaev and Garnier, 1999; Abdullaev and Baizakov, 2000; Abdullaev *et al.*, 2000; Goedde *et al.*, 1997; Kodama and Hasegawa, 1983, 1992; Kodama *et al.*, 1994; Mecozzi *et al.*, 1991; Moores *et al.*, 1994). Stochasticity are basically of two types namely homogenous and nonhomogenous (Elgin, 1993).

1. In the inhomogenous case the stochasticity is present in the input pulse of the fiber. So the parameter dynamics are deterministic but however the initial values are random.
2. In the homogenous case the stochasticity originates due to the random perturbation of the fiber like the density fluctuation of the fiber material or the random variations in the fiber diameter etc.

## 2. MATHEMATICAL FORMULATION

The dimensionless form of the Nonlinear Schrödinger's Equation (NLSE) is given by

$$iq_t + \frac{1}{2}q_{xx} + F(|q|^2)q = 0. \tag{1}$$

where $F$ is a real-valued algebraic function and the smoothness of the complex function $F(|q|^2)q : C \mapsto C$ is necessary. Considering the complex plane $C$ as a two-dimensional linear space $R^2$, the function $F(|q|^2)q$ is $k$ times continously differentiable so that (Biswas, 2003)

$$F(|q|^2)q \in \bigcup_{m,n=1}^{\infty} C^k((-n, n) \times (-m, m); R^2)$$

Equation (1) is a nonlinear partial differential equation (PDE) of parabolic type that is not integrable, in general. The special case, $F(s) = s$, also known as the Kerr law of nonlinearity, in which case it reduces to the cubic Schrödinger's equation, that is integrable by the method of Inverse Scattering Transform (IST) (Hasegawa and Kodama, 1995; Wabnitz *et al.*, 1995). The IST is the nonlinear analog of Fourier transform that is used for solving the linear partial differential equations. Schematically, the IST and the technique of Fourier transform are similar (Hasegawa and Kodama, 1995). The solutions are known as *solitons*. The general case $F(s) \neq s$ takes it away from the IST picture as it is not of Painleve type (Hasegawa and Kodama, 1995). Equation (1), physically, represents the propagation of solitons through an optical fiber.

The three conserved quantities or *integrals of motion* (Biswas, 2003) are the energy $(E)$ or $L_2$ norm, linear momentum $(M)$ and the Hamiltonian $(H)$ that are respectively given by

$$E = \int_{-\infty}^{\infty} |q|^2 dx \tag{2}$$

$$M = \frac{i}{2}\int_{-\infty}^{\infty} (q^* q_x - q q_x^*)\, dx \tag{3}$$

$$H = \int_{-\infty}^{\infty} \left[\frac{1}{2}|q_x|^2 - f(I)\right] dx \tag{4}$$

where

$$f(I) = \int_0^I F(\xi)\, d\xi \tag{5}$$

and the intensity $I$ is given by $I = |q|^2$. The soliton solution of (1), although not integrable, is assumed to be given in the form (Kodama and Ablowitz, 1981)

$$q(x,t) = A(t)g[B(t)(\theta - \bar{\theta}(t))]e^{i\phi(x,t)} \tag{6}$$

where

$$\frac{\partial\theta}{\partial x} = 1, \quad \frac{\partial\theta}{\partial t} = 0, \quad \frac{d\bar{\theta}}{dt} = v \tag{7}$$

with

$$\frac{\partial\phi}{\partial x} = -\kappa \tag{8}$$

and

$$\frac{\partial\phi}{\partial t} = \frac{B^2}{2}\frac{I_{0,0,2}}{I_{0,2,0}} - \frac{\kappa^2}{2} + \frac{1}{I_{0,2,0}}\int_{-\infty}^{\infty} g^2(s)F(A^2g^2(s))\, ds \tag{9}$$

Here, the following integral is defined

$$I_{\alpha,\beta,\gamma} = \int_{-\infty}^{\infty} \tau^{\alpha} g^{\beta}(\tau) \left(\frac{dg}{d\tau}\right)^{\gamma} d\tau \tag{10}$$

for non-negative integers $\alpha$, $\beta$ and $\gamma$ with $\tau = B(t)(\theta - \bar{\theta}(t))$. In (6), $g$ represents the shape of the soliton described by the GNLSE and it depends on the type of nonlinearity in (1). The parameters $A(t)$ and $B(t)$, in (6), respectively represent the soliton amplitude and width, while $\phi(x, t)$ is the phase of the soliton and therefore $K$ is the frequency of the soliton while $v$ is the velocity. The soliton width and the amplitude are related as $B(t) = \lambda(A(t))$ where the functional form $\lambda$ depends on the type of nonlinearity in (1). Also, $\bar{\theta}(t)$ gives the mean position of the soliton. For such a general form of the soliton given by (6), the integrals of motion, from (2), (3) and (4), respectively reduce to

$$E = \int_{-\infty}^{\infty} |q|^2 dx = \frac{A^2}{B} I_{0,2,0} \tag{11}$$

$$M = \frac{i}{2}\int_{-\infty}^{\infty} (q q_x^* - q^* q_x)\, dx = -\kappa \frac{A^2}{B} I_{0,2,0} \tag{12}$$

$$\begin{aligned} H &= \int_{-\infty}^{\infty} \left[\frac{1}{2}|q_x|^2 - f(|q|^2)\right] dx \\ &= \frac{A^2 B}{2} I_{0,0,2} + \frac{\kappa^2 A^2}{2B} I_{0,2,0} - \int_{-\infty}^{\infty}\int_0^I F(s)\, ds dx \end{aligned} \tag{13}$$

For the soliton given by (6), the parameters are now defined as (Biswas, 2003)

$$\kappa(t) = \frac{i}{2} \frac{\int_{-\infty}^{\infty} (q q_x^* - q^* q_x)\, dx}{\int_{-\infty}^{\infty} |q|^2 dx} \tag{14}$$

$$\bar{\theta}(t) = \frac{\int_{-\infty}^{\infty} \theta |q|^2 d\theta}{\int_{-\infty}^{\infty} |q|^2 d\theta} \tag{15}$$

From (11), (14) and (15), the parameter dynamics of the unperturbed soliton is as follows

$$\frac{dE}{dt} = 0 \tag{16}$$

$$\frac{d\kappa}{dt} = 0 \tag{17}$$

$$\frac{d\bar{\theta}}{dt} = -\kappa \tag{18}$$

along with (8) and (9). Here, (9) is obtained by differentiating (6) with respect to $t$ and subtracting from its conjugate while using (1). The parameter dynamics for the amplitude and the width of the soliton individually can be obtained for the special cases of $F(s)$ once the functional form of $F$ is known.

The NLSE along with its perturbation terms is given by

$$iq_t + \frac{1}{2}q_{xx} + F(|q|^2)q = i\varepsilon R[q, q^*] \tag{19}$$

Here $R$ is a spatio-differential operator while the perturbation parameter $\epsilon$, with $0 < \epsilon \ll 1$, represents the relative width of the spectrum in fiber optics that arises due to quasi-monochromaticity (Biswas, 2003). In presence of perturbation terms, as in (19), the integrals of motion are modified. In most instances, a consequence of this is an adiabatic deformation of the soliton parameters like its amplitude, width, frequency and velocity accompanied by small amounts of radiation or small amplitude dispersive waves. The adiabatic parameter dynamics, in presence of perturbation terms, neglecting the radiation, are (Biswas, 2003)

$$\frac{d\kappa}{dt} = \frac{\varepsilon}{I_{0,2,0}}\frac{B}{A^2}\left[i\int_{-\infty}^{\infty}(q_x^* R - q_x R^*)\,dx - \kappa\int_{-\infty}^{\infty}(q^* R + qR^*)\,dx\right] \tag{20}$$

$$\frac{d\bar{\theta}}{dt} = -\kappa + \frac{\varepsilon}{I_{0,2,0}}\frac{B}{A^2}\int_{-\infty}^{\infty} x(q^* R + qR^*)\,dx \tag{21}$$

$$\frac{\partial\phi}{\partial t} = \frac{B^2}{2}\frac{I_{0,0,2}}{I_{0,2,0}} - \frac{\kappa^2}{2} + \frac{1}{I_{0,2,0}}\int_{-\infty}^{\infty} g^2(s)F(A^2 g^2(s))\,ds + \frac{i\varepsilon}{I_{0,2,0}}\frac{B}{2A^2}\int_{-\infty}^{\infty}(qR^* - q^* R)\,dx \tag{22}$$

## 3. POWER LAW NONLINEARITY

Power law nonlinearity is exhibited in various materials including semiconductors. This law also occurs in media for which higher order photon processes dominate at different intensities. This law is also treated as a generalization to the Kerr law nonlinearity.

For the case of power law nonlinearity, $F(s) = s^p$ so that $f(s) = s^{p+1}/(p+1)$ so that the NLSE given by (1) modifies to

$$iq_t + \frac{1}{2}q_{xx} + |q|^{2p}q = 0 \tag{23}$$

In (23), it is necessary to have $0 < p < 2$ to prevent wave collapse (Biswas, 2003, 2004) and, in particular, $p \neq 2$ to avoid self-focussing singularity (Abdullaev and

Garnier, 1999). The soliton solution of (23) is given by (Biswas, 2003)

$$q(x,t) = \frac{A}{\cosh^{\frac{1}{p}}[B(x-\bar{x}(t))]} e^{i(-\kappa x+\omega t+\sigma_0)} \tag{24}$$

where

$$\kappa = -v \tag{25}$$

and

$$\omega = \frac{B^2}{2p^2} - \frac{\kappa^2}{2} \tag{26}$$

while

$$B = A^p \left(\frac{2p^2}{1+p}\right)^{\frac{1}{2}} \tag{27}$$

On comparing (24) with (6),

$$g(\tau) = \frac{1}{\cosh^{\frac{1}{p}} \tau} \tag{28}$$

while the phase is given by

$$\phi(x,t) = -\kappa t + \omega t + \sigma_0 \tag{29}$$

where $\omega$ is the wave number and $\sigma_0$ is the center of phase of the soliton.

Considering the effects of perturbation (Blow *et al.*, 1988; Kaup, 1990) on the propagation of solitons through optical fibers, (23) is modified to

$$iq_t + \frac{1}{2}q_{xx} + |q|^{2p}q = i\varepsilon R \tag{30}$$

where

$$R = \delta|q|^{2m}q + \beta q_{xx} - \gamma q_{xxx} + \lambda(|q|^2 q)_x + \nu(|q|^2)_x q + \sigma(x,t) \tag{31}$$

For the perturbation terms, in (31), $\delta < 0$ is the nonlinear damping coefficient (Blow *et al.*, 1988), $\beta$ is the bandpass filtering term (Kaup, 1990; Wabnitz *et al.*, 1995). Also, $\lambda$ is the self-steepening coefficient for short pulses (Hasegawa and Kodama, 1995; Kodama and Hasegawa, 1992) (typically $\leq$100 femto seconds), $\nu$ is the higher order dispersion coefficient (Hasegawa and Kodama, 1995; Wabnitz *et al.*, 1995) and $\gamma$ is the coefficient of the third order dispersion (Hasegawa and Kodama, 1995; Kodama and Hasegawa, 1992; Wabnitz *et al.*, 1995).

It is known that the NLSE, as given by (23), does not give correct prediction for pulse widths smaller than 1 picosecond. For example, in solid state solitary lasers, where pulses as short as 10 femtoseconds are generated, the approximation breaks down. Thus, quasi-monochromaticity is no longer valid and so higher order dispersion terms come in. If the group velocity dispersion is close to zero, one needs to consider the third order dispersion for performance enhancement along trans-oceanic distances. Also, for short pulse widths where group velocity dispersion changes, within the spectral bandwidth of the signal cannot be neglected, one needs to take into account the presence of the third order dispersion (Hasegawa and Kodama, 1995; Potasek, 1989). The perturbation terms due to $\alpha$ and $\beta$ are of non Hamiltonian or non conservative type while those due to $\lambda$, $\nu$ and $\gamma$ are of Hamiltonian or conservative type.

The amplifiers, although needed to restore the soliton energy, introduces noise originating from amplified spontaneous emmision (ASE). To study the impact of noise on soliton evolution, the evolution of the mean free velocity of the soliton due to ASE will be studied in this paper. In case of lumped amplification, solitons are perturbed by ASE in a discrete fashion at the location of the amplifiers. It can be assumed that noise is distributed all along the fiber length since the amplifier spacing satisfies $z_a \ll 1$ (Kivshar and Agarwal, 2003). In (8), $\sigma(x, t)$ represents the Markovian stochastic process with Gaussian statistics and is assumed that $\sigma(x, t)$ (Biswas, 2004; Elgin, 1993) is a function of $t$ only so that $\sigma(x, t) = \sigma(t)$. Now, the complex stochastic term $\sigma(t)$ can be decomposed into real and imaginary parts as

$$\sigma(t) = \sigma_1(t) + i\sigma_2(t) \tag{32}$$

is further assumed to be independently delta correlated in both $\sigma_1(t)$ and $\sigma_2(t)$ with

$$\langle\sigma_1(t)\rangle = \langle\sigma_2(t)\rangle = \langle\sigma_1(t)\sigma_2(t')\rangle = 0 \tag{33}$$

$$\langle\sigma_1(t)\sigma_1(t')\rangle = 2D_1\delta(t - t') \tag{34}$$

$$\langle\sigma_2(t)\sigma_2(t')\rangle = 2D_2\delta(t - t') \tag{35}$$

where $D_1$ and $D_2$ are related to the ASE spectral density. In this paper, it is assumed that $D_1 = D_2 = D$. Thus,

$$\langle\sigma(t)\rangle = 0 \tag{36}$$

and

$$\langle\sigma(t)\sigma(t')\rangle = 2D\delta(t - t') \tag{37}$$

In soliton units, one gets,

$$D = \frac{F_n - F_G}{N_{ph} z_a} \tag{38}$$

where $F_n$ is the amplifier noise figure, while

$$F_G = \frac{(G-1)^2}{G \ln G} \tag{39}$$

is related to the amplifier gain $G$ and finally $N_{ph}$ is the average number of photons in the pulse propagating as a fundamental soliton.

### 3.1. Mathematical Analysis

The three integrals of motion of the power law nonlinearity are

$$E = \int_{-\infty}^{\infty} |q|^2 dx$$
$$= A^{2-p}\left(\frac{1+p}{2p^2}\right)^{\frac{1}{2}} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{p}\right)}{\Gamma\left(\frac{1}{p}+\frac{1}{2}\right)} = B^{\frac{2-p}{p}}\left(\frac{1+p}{2p^2}\right)^{\frac{1}{p}} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{p}\right)}{\Gamma\left(\frac{1}{p}+\frac{1}{2}\right)} \quad (40)$$

$$M = \frac{i}{2}\int_{-\infty}^{\infty} (q^* q_x - q q_x^*)\, dx$$
$$= 2\kappa A^{2-p}\left(\frac{1+p}{2p^2}\right)^{\frac{1}{2}} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{p}\right)}{\Gamma\left(\frac{1}{p}+\frac{1}{2}\right)}$$
$$= 2\kappa B^{\frac{2-p}{p}}\left(\frac{1+p}{2p^2}\right)^{\frac{1}{p}} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{p}\right)}{\Gamma\left(\frac{1}{p}+\frac{1}{2}\right)} \quad (41)$$

and

$$H = \int_{-\infty}^{\infty}\left[\frac{1}{2}|q_x|^2 - \frac{1}{p+1}|q|^{2p+2}\right] dx$$
$$= \frac{B^{\frac{2}{p}}}{2p^2}\left(\frac{1+p}{2p^2}\right)^{\frac{1}{p}}\left[\frac{(B^2+\kappa^2p^2)}{B}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{p}\right)}{\Gamma\left(\frac{1}{p}+\frac{1}{2}\right)} - 2B\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\frac{p+1}{p}}{\Gamma\left(\frac{p+1}{p}+\frac{1}{2}\right)}\right]$$
$$= \frac{A^2}{2p^2}\left[\left\{A^p\left(\frac{2p^2}{1+p}\right)^{\frac{1}{p}} + \frac{\kappa^2p^2}{A^p}\left(\frac{1+p}{2p^2}\right)^{\frac{1}{2}}\right\}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{p}\right)}{\Gamma\left(\frac{1}{p}+\frac{1}{2}\right)}\right.$$
$$\left. - 2A^p\left(\frac{2p^2}{1+p}\right)^{\frac{1}{2}}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{p+1}{p}\right)}{\Gamma\left(\frac{p+1}{p}+\frac{1}{2}\right)}\right] \quad (42)$$

Using the first two integrals of motion, one can write

$$\frac{dA}{dt} = \frac{\varepsilon}{2-p} A^{p-1} \left(\frac{2p^2}{1+p}\right)^{\frac{p-1}{2p}} \frac{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{p}\right)} \int_{-\infty}^{\infty} (q^* R + q R^*)\, dx \quad (43)$$

$$\frac{d\kappa}{dt} = \varepsilon B^{\frac{p-2}{p}} \left(\frac{2p^2}{1+p}\right)^{\frac{1}{p}} \frac{\Gamma\left(\frac{1}{p}+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{p}\right)} \left[ i \int_{-\infty}^{\infty} (q_x^* R - q_x R^*)\, dx - \kappa \int_{-\infty}^{\infty} (q^* R + q R^*)\, dx \right] \quad (44)$$

Now substituting the perturbation terms $R$ from (31) and carrying out the integrations in (43) and (44) yields

$$\begin{aligned}\frac{dA}{dt} &= \frac{2\varepsilon\delta}{2-p} A^{2m+1} \left(\frac{1+p}{2p^2}\right)^{\frac{1}{2p}} \frac{\Gamma\left(\frac{1}{p}+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{p}\right)} \frac{\Gamma\left(\frac{m+1}{p}\right)}{\Gamma\left(\frac{m+1}{p}+\frac{1}{2}\right)} \\ &\quad + \frac{2\varepsilon\beta}{2-p} \frac{A^{p-1}}{B} \left(\frac{2p^2}{p+1}\right)^{\frac{p-1}{2p}} \frac{\Gamma\left(\frac{1}{p}+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{p}\right)} \\ &\quad \left[ \frac{A^2B^2}{p^2} \frac{\Gamma\left(\frac{p+1}{p}\right)}{\Gamma\left(\frac{p+1}{p}+\frac{1}{2}\right)} - \frac{A^2}{p^2}(\kappa^2 p^2 + B^2) \frac{\Gamma\left(\frac{1}{p}\right)}{\Gamma\left(\frac{1}{p}+\frac{1}{2}\right)} \right] \\ &\quad + \frac{2\varepsilon A}{2-p} A^{p-1} \left(\frac{2p^2}{1+p}\right)^{\frac{p-1}{2p}} \frac{\Gamma\left(\frac{1}{p}+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{p}\right)} \left[ \sigma_1 \int_{-\infty}^{\infty} \frac{\cos\phi}{\cosh^{\frac{1}{p}}\tau} dx \right. \\ &\quad \left. + \sigma_2 \int_{-\infty}^{\infty} \frac{\sin\phi}{\cosh^{\frac{1}{p}}\tau} dx \right] \end{aligned} \quad (45)$$

$$\begin{aligned}\frac{d\kappa}{dt} &= \frac{4\varepsilon\beta}{p^2} \kappa A^2 B^{\frac{2p-2}{p}} \left(\frac{2p^2}{p+1}\right) \left(\frac{p-2}{p+2}\right) \\ &\quad + 4\varepsilon\kappa A B^{\frac{p-2}{p}} \left(\frac{2p^2}{1+p}\right)^{\frac{1}{p}} \frac{\Gamma\left(\frac{1}{p}+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)} \int_{-\infty}^{\infty} \left[ \frac{B}{p} \frac{\tanh\tau}{\cosh^{\frac{1}{p}}\tau} (\sigma_2\cos\phi - \sigma_1\sin\phi) \right. \\ &\quad \left. + \frac{2\kappa}{\cosh^{\frac{1}{p}}\tau} (\sigma_1\cos\phi + \sigma_2\sin\phi) \right] dx \end{aligned} \quad (46)$$

Equations (45) and (46), as it appears, is difficult to analyse. If the terms with $\sigma_1$ and $\sigma_2$ are suppressed, the resulting dynamical system has a stable fixed point, namely a sink, given by $(\bar{A}, \bar{\kappa}) = (\bar{A}, 0)$ where

$$\bar{A} = \left[ \frac{2\beta}{\delta(p+1)} \frac{\Gamma\left(\frac{m+1}{p}+\frac{1}{2}\right)}{\Gamma\left(\frac{m+1}{p}\right)} \left\{ \frac{\Gamma\left(\frac{p+1}{p}\right)}{\Gamma\left(\frac{p+1}{p}+\frac{1}{2}\right)} - \frac{\Gamma\left(\frac{1}{p}\right)}{\Gamma\left(\frac{1}{p}+\frac{1}{2}\right)} \right\} \right]^{\frac{1}{2(m-p)}} \tag{47}$$

This phenomenon, known as *soliton cooling* (Blow *et al.*, 1988; Hasegawa and Kodama, 1995), is used to lock the frequency and the amplitude of the soliton to a fixed value for the stable propagation of solitons through optical fibers [26]. Now, linearizing the dynamical system about this fixed point gives, after simplification

$$\frac{dA}{dt} = -\varepsilon \left( A^{2m+1} - \frac{\xi}{A} \right) \tag{48}$$

$$\frac{d\kappa}{dt} = -\varepsilon[\kappa - \zeta(1 + A - \kappa)] \tag{49}$$

where

$$\xi = \sigma_1 \int_{-\infty}^{\infty} \frac{\cos\phi}{\cosh^{\frac{1}{p}} \tau} dx + \sigma_2 \int_{-\infty}^{\infty} \frac{\sin\phi}{\cosh^{\frac{1}{p}} \tau} dx \tag{50}$$

and

$$\zeta = \int_{-\infty}^{\infty} \left[ \frac{B}{p} \frac{\tanh\tau}{\cosh^{\frac{1}{p}} \tau} (\sigma_2 \cos\phi - \sigma_1 \sin\phi) + \frac{2\kappa}{\cosh^{\frac{1}{p}} \tau} (\sigma_1 \cos\phi + \sigma_2 \sin\phi) \right] dx \tag{51}$$

Equations (48) and (49) are called the *Langevin* equations which will now be analyzed to compute the soliton mean drift velocity of the soliton. If the soliton parameters are chosen such that $\zeta A$ is small, then (49) gives

$$\frac{d\kappa}{dt} = -\varepsilon\left[\kappa - \zeta(1 - \kappa)\right] \tag{52}$$

One can solve (52) for $\kappa$ and eventually the mean drift velocity of the soliton can be obtained. The stochastic phase factor of the soliton is defined by

$$\psi(t, y) = \int_{y}^{t} \zeta(s)\, ds \tag{53}$$

where $t > y$. Assuming that $\sigma$ is a Gaussian stochastic variable we arrive at

$$\langle e^{\psi(t,y)} \rangle = e^{D(t-y)} \tag{54}$$

$$\langle e^{[\psi(t,y)+\psi(t',y')]}\rangle = e^{D\theta} \tag{55}$$

where

$$\theta = 2(t + t' - y - y') - |t - t'| - |y - y'| \tag{56}$$

and

$$\langle \zeta(y)e^{-\psi(t,y)}\rangle = \frac{\partial}{\partial y}\langle e^{-\psi(t,y)}\rangle = De^{D(t-y)} \tag{57}$$

$$\langle \zeta(y)\zeta(y')e^{[-\psi(t,y)-\psi(t',y')]}\rangle = 2D\delta(y - y')e^{D\theta} + \frac{\partial^2}{\partial y\partial y'}e^{D\theta} \tag{58}$$

Now solving (49) with the initial condition as $\kappa(0) = 0$ and using equations (53) to (58) the soliton mean drift velocity is given by

$$\langle \kappa(t)\rangle = -\frac{D}{1 - D}\left\{1 - e^{-\varepsilon(1-D)t}\right\} \tag{59}$$

From (59), it follows that

$$\lim_{t\to\infty}\langle \kappa(t)\rangle = \frac{D}{1 - D} \tag{60}$$

Thus, for large $t$, $\langle \kappa(z)\rangle$ approaches a constant value provided $D < 1$. Thus, the soliton mean frequency and hence the mean drift velocity of the soliton, approaches a constant for a large time. For $D > 1$, $\langle \kappa(t)\rangle$ becomes unbounded for large $t$.

## 4. CONCLUSIONS

In this paper, the dynamics of optical solitons with power law nonlinearity in presence of perturbation terms, both deterministic as well as of stochastic, are studied. The Langevin equations were derived and the corresponding parameter dynamics was studied. The mean drift velocity of the soliton was obtained.

In this study, it was assumed that the stochastic perturbation term $\sigma$ is a function of $t$ only, for simplicity. However, in reality $\sigma$ is a function of both $x$ and $t$ and thus making it a far more difficult system to analyze although such kind of situations are being presently studied. Although, in this paper the stochastic perturbation due to other non-Kerr law nonlinearities was not studied, the results of those studies are awaited at this time.

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